

Bounds on the Concavity of Quantum Entropy

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Dedicated to the memory of Oscar E. Lanford III

Abstract

We give new upper and lower bounds on the concavity of quantum entropy. Comparisons are given with other results in the literature.

1 Introduction

It is well-known that the von Neumann quantum entropy $S(\rho) \equiv -\text{Tr } \rho \log \rho$ is concave, where the argument ρ is a density matrix ρ , i.e., $\rho \geq 0$, $\text{Tr } \rho = 1$. Recently Kim [8] proved the following lower bound on the concavity

$$S(\rho_{\text{Av}}) - xS(\rho_1) - (1-x)S(\rho_2) \geq \frac{x(1-x)}{(1-2x)^2} \max \begin{cases} H(\rho_{\text{Av}}, \rho_{\text{Rev}}) \\ H(\rho_{\text{Rev}}, \rho_{\text{Av}}) \end{cases} \quad (1)$$

$$\geq \frac{1}{2}x(1-x)\|\rho_1 - \rho_2\|_1^2 \quad (2)$$

where $\rho_{\text{Av}} \equiv x\rho_1 + (1-x)\rho_2$, $\rho_{\text{Rev}} \equiv x\rho_2 + (1-x)\rho_1$ and $H(\rho, \gamma) \equiv \text{Tr } \rho(\log \rho - \log \gamma)$ denotes the relative entropy. In Section 2 present a simple proof of (2) and compare it to other possible lower bounds, particularly those resulting from recent work of Carlen and

Lieb [4]. Our results demonstrate that their methods do not always give stronger results than those obtained using Pinsker's bound. We also consider upper bounds in Section 3. Finally, in section 4 we discuss the possibility of improved bounds using variants of the Renyi relative entropy.

2 Lower bounds

We begin with a simple proof of (2)

$$\begin{aligned} S(\rho_{Av}) - xS(\rho_1) - (1-x)S(\rho_2) &= x\text{Tr } \rho_1 \log \rho_1 + (1-x)\text{Tr } \rho_2 \log \rho_2 - \text{Tr } [x\rho_1 + (1-x)\rho_2] \log \rho_{Av} \\ &= xH(\rho_1, \rho_{Av}) + (1-x)H(\rho_2, \rho_{Av}) \end{aligned} \quad (3)$$

$$\begin{aligned} &\geq \frac{1}{2}x\|\rho_1 - \rho_{Av}\|_1^2 + \frac{1}{2}(1-x)\|\rho_2 - \rho_{Av}\|_1^2 \\ &= x(1-x)\frac{1}{2}\|\rho_1 - \rho_2\|_1^2 \end{aligned} \quad (4)$$

where we used Pinsker's bound [14] [13, Theorem 1.15]

$$H(\rho, \gamma) \geq \frac{1}{2}\|\rho - \gamma\|_1^2 \quad (5)$$

In view of the fact that Carlen-Lieb [4] presented an inequality for subadditivity that can be stronger than Pinsker's bound in some situations, it is interesting to see if one can use their results to improve the bound (2). We begin with the well-known observation that if

$$P_{AB} = \begin{pmatrix} x\rho_1 & 0 \\ 0 & (1-x)\rho_2 \end{pmatrix} \quad (6)$$

so that $P_A = \rho_{Av}$ and $P_B = \begin{pmatrix} x & 0 \\ 0 & 1-x \end{pmatrix}$, then

$$\begin{aligned} H(P_{AB}, P_A \otimes P_B) &= S(P_A) + S(P_B) - S(P_{AB}) \\ &= S(\rho_{Av}) - xS(\rho_1) - (1-x)S(\rho_2) \end{aligned} \quad (7)$$

It then follows from [4, Lemma 2.1] that

$$\begin{aligned} S(\rho_{Av}) - xS(\rho_1) - (1-x)S(\rho_2) &\geq -2 \log \left[1 - \frac{1}{2} \text{Tr} \left(\sqrt{P_{AB}} - \sqrt{P_A \otimes P_B} \right)^2 \right] \end{aligned} \quad (8)$$

$$\begin{aligned} &= -2 \log \left[1 - \frac{1}{2} \text{Tr} \left(P_{AB} + P_A \otimes P_B - 2\sqrt{P_{AB}}\sqrt{P_A \otimes P_B} \right) \right] \\ &= -2 \log \text{Tr} \sqrt{P_{AB}}\sqrt{P_A \otimes P_B} \end{aligned} \quad (9)$$

$$= -2 \log \text{Tr} \begin{pmatrix} x\sqrt{\rho_1}\sqrt{\rho_{Av}} & 0 \\ 0 & (1-x)\sqrt{\rho_2}\sqrt{\rho_{Av}} \end{pmatrix}$$

$$\begin{aligned}
&= -2 \log \text{Tr} [x\sqrt{\rho_1} + (1-x)\sqrt{\rho_2}] \sqrt{\rho_{\text{Av}}} \\
&\geq -2 \log \text{Tr} \sqrt{\rho_{\text{Av}}} \sqrt{\rho_{\text{Av}}} = 0
\end{aligned} \tag{10}$$

where the last inequality uses the fact that $f(u) = \sqrt{u}$ is a concave operator function. Note that the bound (9) could have been obtained directly from the monotonicity of the Renyi relative entropy

$$H_a^{\text{Ren}}(\rho, \gamma) \equiv \frac{1}{a-1} \log \text{Tr} \rho^a \gamma^{1-a} \tag{11}$$

in the case

$$H(\rho, \gamma) = H_1^{\text{Ren}}(\rho, \gamma) \geq H_{1/2}^{\text{Ren}}(\rho, \gamma) \tag{12}$$

If one applies Pinsker's inequality to (7), one would obtain

$$S(\rho_{\text{Av}}) - xS(\rho_1) - (1-x)S(\rho_2) = H(P_{AB}, P_A \otimes P_B) \geq 2x^2(1-x)^2 \tag{13}$$

which appears to be weaker than (9).

This raises the question of which of the lower bounds on concavity is stronger. Numerical tests show that in some cases (2) is stronger and in other cases (9) is stronger. (See the Appendix for specific examples.) Thus the Carlen-Lieb bound [4] on subadditivity does not give a stronger bound on the concavity of entropy than Pinsker's inequality.

3 Upper bounds

It is also a well-known consequence [5, 9] of the operator monotonicity of the function $f(u) = \log u$ that the concavity satisfies the upper bound

$$S(\rho_{\text{Av}}) - xS(\rho_1) - (1-x)S(\rho_2) \leq -x \log x - (1-x) \log(1-x) \equiv h(x) \tag{14}$$

where $h(x)$ denotes the binary entropy¹ [3, 16]

$$D_{\text{Bures}}^2(\rho, \gamma) \equiv 2 \left[1 - \text{Tr} (\sqrt{\rho} \gamma \sqrt{\rho})^{1/2} \right] \leq \|\rho - \gamma\|_1 \tag{15}$$

with the inequality due to Fuchs and van de Graaf [6], the inequality

$$S(\rho_{\text{Av}}) - xS(\rho_1) - (1-x)S(\rho_2) \leq h(x) D_{\text{Bures}}^2(\rho_1, \rho_2) \leq h(x) \|\rho_1 - \rho_2\|_1 \tag{16}$$

was proved in [15]. However, because one can have $\|\rho_1 - \rho_2\|_1 > 1$, (16) is not always stronger than (14).

Recently Audenaert [1, Eq. 66] obtained the stronger upper bound

$$S(\rho_{\text{Av}}) - xS(\rho_1) - (1-x)S(\rho_2) \leq h(x) \frac{1}{2} \|\rho_1 - \rho_2\|_1 \tag{17}$$

which is also stronger than (14).

A different upper bound is given in Theorem 3 below.

¹ $\text{Tr} (\sqrt{\rho} \gamma \sqrt{\rho})^{1/2}$ is known as the fidelity in quantum information theory.

4 Renyi bounds

We can summarize the basic bounds as

Theorem 1 *The quantum entropy satisfies the upper and lower bounds*

$$\begin{aligned} h(x)\frac{1}{2}\|\rho_1 - \rho_2\|_1 &\geq S(\rho_{\text{Av}}) - xS(\rho_1) - (1-x)S(\rho_2) \\ &\geq \frac{1}{2}x(1-x)\|\rho_1 - \rho_2\|_1^2 \end{aligned} \quad (18)$$

It is natural to ask if one can use the monotonicity of either the Renyi relative entropy (11) or the sandwiched Renyi entropy

$$\tilde{H}_a^{\text{Ren}}(\rho, \gamma) \equiv \text{Tr} \frac{1}{a-1} \log \text{Tr} \left(\gamma^{\frac{1-a}{2a}} \rho \gamma^{\frac{1-a}{2a}} \right)^a \quad (19)$$

introduced independently in [12] and [17] to improve these bounds. It is well-known that the Renyi relative entropy (11) is monotone in a (see, e.g., [10, 12]) and Beigi [2] (see also [12]) recent proved that this monotonicity also holds for the sandwiched Renyi entropy (19) when $a \geq \frac{1}{2}$. Because $\tilde{H}_a^{\text{Ren}}(\rho, \gamma) \leq H_a^{\text{Ren}}(\rho, \gamma) \quad \forall a$, it is always more advantageous to use \tilde{H}_a^{Ren} for upper bounds and H_a^{Ren} for lower bounds. Since

$$\lim_{a \rightarrow 1} \tilde{H}_a^{\text{Ren}}(\rho, \gamma) = H(\rho, \gamma) = \lim_{b \rightarrow 1} H_b^{\text{Ren}}(\rho, \gamma) \quad (20)$$

it follows that

Theorem 2 *For any fixed, $x \in (0, 1)$ and $\rho_1 \neq \rho_2$ one can find $a_c > 1$ and $b_c \in [\frac{1}{2}, 1)$ such that*

$$\begin{aligned} h(x)\frac{1}{2}\|\rho_1 - \rho_2\|_1 &\geq x\tilde{H}_a^{\text{Ren}}(\rho_1, \rho_{\text{Av}}) + (1-x)\tilde{H}_a^{\text{Ren}}(\rho_2, \rho_{\text{Av}}) \quad \forall a \geq a_c \\ &\geq S(\rho_{\text{Av}}) - xS(\rho_1) - (1-x)S(\rho_2) \\ &\geq xH_b^{\text{Ren}}(\rho_1, \rho_{\text{Av}}) + (1-x)H_b^{\text{Ren}}(\rho_2, \rho_{\text{Av}}) \quad \forall b \in [b_c, 1) \\ &\geq \frac{1}{2}x(1-x)\|\rho_1 - \rho_2\|_1^2 \end{aligned}$$

Except for very special situations, e.g., both ρ_1, ρ_2 multiples of orthogonal projections, one would expect that one can find a_c, b_c such that all of the above bounds are strict. However, it seems unlikely that there exist a and/or b which improve the basic bounds in Theorem 1 for arbitrary x, ρ_1, ρ_2 . Whether or not one can obtain improved bounds if some of these parameters are fixed seems to be an open question in general.

In the special case $x = \frac{1}{2}$, observe that

$$\begin{aligned} \frac{1}{2} \left[H_2^{\text{Ren}}(\rho_1, \rho_{\text{Av}}) + H_2^{\text{Ren}}(\rho_2, \rho_{\text{Av}}) \right] &= \frac{1}{2} \left[\log \text{Tr} \rho_1^2 \rho_{\text{Av}}^{-1} + \log \text{Tr} \rho_2^2 \rho_{\text{Av}}^{-1} \right] \\ &\leq \frac{1}{2} \left[\log(2 \text{Tr} \rho_1) + \log(2 \text{Tr} \rho_2) \right] = \log 2 \end{aligned} \quad (21)$$

where we used for $k = 1, 2$

$$\mathrm{Tr} \rho_k^2 \rho_{\mathrm{Av}}^{-1} = 2 \mathrm{Tr} \rho_k^{1/2} \left(\rho_k^{1/2} \frac{1}{\rho_1 + \rho_2} \rho_k^{1/2} \right) \rho_k^{1/2} \leq 2 \mathrm{Tr} \rho_k^{1/2} I \rho_k^{1/2} = 2 \mathrm{Tr} \rho_k \quad (22)$$

and the inequalities are strict if $\rho_1 \neq \rho_2$ and they are not multiples of pairwise orthogonal projections. Then we can conclude that for $x = \frac{1}{2}$,

$$\begin{aligned} h(\tfrac{1}{2}) = \log 2 &\geq \tfrac{1}{2} \left[H_a^{\mathrm{Ren}}(\rho_2, \rho_{\mathrm{Av}}) + H_a^{\mathrm{Ren}}(\rho_1, \rho_{\mathrm{Av}}) \right] \\ &\geq \tfrac{1}{2} \left[\tilde{H}_a^{\mathrm{Ren}}(\rho_1, \rho_{\mathrm{Av}}) + \tilde{H}_a^{\mathrm{Ren}}(\rho_2, \rho_{\mathrm{Av}}) \right] \\ &\geq S(\rho_{\mathrm{Av}}) - xS(\rho_1) - (1-x)S(\rho_2) \end{aligned} \quad (23)$$

for all $a \in (1, 2]$. This led us to conjecture that a similar result could be proved for any fixed x . Milan Mosonyi [11] then proved the following slightly stronger result and kindly allowed us to include his argument here.

Theorem 3 (Mosonyi) *For all $a > 1$,*

$$\begin{aligned} h(x) &\geq x \left[\tilde{H}_a^{\mathrm{Ren}}(\rho_1, \rho_{\mathrm{Av}}) + (1-x) \tilde{H}_a^{\mathrm{Ren}}(\rho_2, \rho_{\mathrm{Av}}) \right] \\ &\geq S(\rho_{\mathrm{Av}}) - xS(\rho_1) - (1-x)S(\rho_2) \end{aligned} \quad (24)$$

Proof: Note that the sandwiched Renyi divergences are monotone increasing in the parameter a , and their limit when $a \rightarrow \infty$ is the max-relative entropy

$$H_{\max}(\rho, \gamma) := \inf \{ \log \omega : \rho \leq \omega \gamma \}.$$

Obviously, $\rho_1 \leq (1/x)\rho_{\mathrm{Av}}$, and hence

$$\tilde{H}_a(\rho_1, \rho_{\mathrm{Av}}) \leq H_{\max}(\rho_1, \rho_{\mathrm{Av}}) \leq -\log x \quad \forall a \in [1/2, +\infty].$$

Applying the same result to ρ_2 , shows that (24) holds for every $a > 1$.

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A Numerical Examples

Since all of the numerical examples were found for qubit density matrices, it is convenient to represent them using the Bloch sphere representation $\rho = \frac{1}{2}[I + \mathbf{w} \cdot \boldsymbol{\sigma}] \equiv \frac{1}{2}[I + \sum_k w_k \sigma_k]$ identify \mathbf{w}_k with ρ_k .

$$\mathbf{w}_1 = (0.2876, 0.4322, 0.3112) \quad \mathbf{w}_2 = (-0.1552, -0.0532, -0.0874) \quad x = 0.7086 \quad (25a)$$

$$\mathbf{w}_1 = (-0.2136, 0.0702, -0.0944) \quad \mathbf{w}_2 = (-0.5204, 0.7790, -0.1772) \quad x = 0.2197 \quad (25b)$$

$$\mathbf{w}_1 = (-0.1850, 0.7506, -0.6388) \quad \mathbf{w}_2 = (0.0254, 0.0012, 0.0114) \quad x = 0.5218 \quad (25c)$$

For Example (a), (2) yields a better bound than (9). For Example (b), (9) yields a better bound (2). It is interesting that even (1), which is stronger than (2), is sometimes weaker than (9). This is the case for Example (c) for which (9) is strictly greater than (1).

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